EXERCISES [MAI 5.9]

RELATED RATES

SOLUTIONS

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A. Paper 1 questions (SHORT)

1. (a) $C = 2A^3 + 1 \Rightarrow \frac{dC}{dt} = 6A^2 \frac{dA}{dt}$

$\Rightarrow \frac{dC}{dt} = 18A^2$.

When $A = 2$, then $\frac{dC}{dt} = 72$.

(b) $\ln D = \frac{3}{B} \Rightarrow \frac{1}{D}\frac{dD}{dt} = -\frac{3}{B^2}\frac{dB}{dt}$

$\Rightarrow \frac{dD}{dt} = -\frac{6D}{B^2}$.

When $D = e, B = 3$ then $\frac{dD}{dt} = -\frac{2e}{3}$

2. $F = 2A^2B + 2B^3 \Rightarrow \frac{dF}{dt} = 4AB\frac{dA}{dt} + 2A^2\frac{dB}{dt} + 6B^2\frac{dB}{dt}$

$\frac{dF}{dt} = 12AB + 4A^2 + 12B^2$.

When $A = B = 1$ then $\frac{dF}{dt} = 28$

3. $F^4 = 2A^2B + 2B^3 \Rightarrow 4F^3\frac{dF}{dt} = 4AB\frac{dA}{dt} + 2A^2\frac{dB}{dt} + 6B^2\frac{dB}{dt}$

$\Rightarrow 2F^3\frac{dF}{dt} = 2AB\frac{dA}{dt} + A^2\frac{dB}{dt} + 3B^2\frac{dB}{dt}$

Thus, $\Rightarrow 2F^3\frac{dF}{dt} = 6AB + 2A^2 + 6B^2$

When $A = B = 1$ then $F^4 = 4 \Rightarrow F^2 = 2 \Rightarrow F = \sqrt{2}$

Finally,

$\Rightarrow 2\sqrt{2}^3\frac{dF}{dt} = 6 + 2 + 6 \Rightarrow 4\sqrt{2}\frac{dF}{dt} = 14 \Rightarrow 4\sqrt{2}\frac{dF}{dt} = \frac{7}{2\sqrt{2}}$

4. $\frac{dV}{dt} = 8(\text{cm}^3\text{s}^{-1})$,

$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \left(\frac{dV}{dt}\right) \div \left(\frac{dV}{dr}\right)$

When $r= 2$, $\frac{dr}{dt} = 8 \div (4\pi\times 2^2) = \frac{1}{2\pi}(\text{cm s}^{-1})$ (i.e. $\cong 0.159$)

5. Let h = height of triangle and $\theta = \hat{CAB}$.

$h = 5\tan\theta \Rightarrow \frac{dh}{dt} = \frac{5}{\cos^2 x}\frac{d\theta}{dt}$

Put $\theta = \frac{\pi}{3}$. $2 = 5 \times 4 \times \frac{d\theta}{dt}$

$\frac{d\theta}{dt} = \frac{1}{10}$ rad per sec $\left(\text{Accept } \frac{18°}{\pi} \text{ per second or } 5.73° \text{ per second}\right)$

6. $\tan\theta = \frac{3}{x} \Rightarrow \frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{-3}{x^2}\frac{dx}{dt}$

when $\theta = \frac{\pi}{3}$, $x^2 = 3$ and $\cos^2\theta = \frac{1}{4}$

Hence, $4\frac{1}{60} = -\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -\frac{1}{15}$ km s^{-1}= –240 km h^{-1}

The aeroplane is moving towards him at 240 km h^{-1}

7. If z is the distance then $z^2 = 3^2 + x^2 \Rightarrow z^2 = 9 + x^2$

Then $2z\frac{dz}{dt} = 2x\frac{dx}{dt} \Rightarrow z\frac{dz}{dt} = x\frac{dx}{dt}$

When $\theta = \frac{\pi}{3}$, $x = \sqrt{3}$, $z = \sqrt{12} = 2\sqrt{3}$ and $\frac{dx}{dt} = -240$ km h^{-1}

Hence,

$\Rightarrow 2\sqrt{3}\frac{dz}{dt} = -240\sqrt{3} \Rightarrow \frac{dz}{dt} = -120$ km h^{-1}

8.

$\frac{r}{h} = \frac{4}{12} \Rightarrow r = \frac{h}{3}$

$V = \frac{\pi}{3}\left(\frac{h}{3}\right)^2(h)$

$V = \frac{\pi}{27}h^3$

$\frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}$

$10 = \frac{\pi}{9}(6)^2\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{90}{36\pi}\left(= \frac{5}{2\pi} = 0.796\right)$ metres per min

9.

$z^2 = x^2 + y^2$ (or equivalent)

$z = \sqrt{0.8^2 + 0.6^2}$ (=1, initially)

Attempting to differentiate implicitly with respect to t

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$\frac{dz}{dt} = -(0.8\times 60) - (0.6\times 70)$

Rate is -90 (kmh^{-1})

10. Then $\frac{dx}{dt} = 60$ (positive) , $\frac{dz}{dt} = +0.8\times 60 - 0.6\times 70 = 6kmh^{-1}$

11.

$81\pi = \frac{4}{3}\pi(3)^3 + \pi(3)^2 h$

$81\pi = 36\pi + 9\pi h$

$h = 5$ (cm)

$V = \frac{4}{3}\pi r^3 + \pi r^2 h$

Attempt to differentiate with respect to time.

$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} +$

$2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$

$204\pi = 4\pi(3)^2(2) + 2\pi(3)(5)(2) + \pi(3)^2\frac{dh}{dt}$

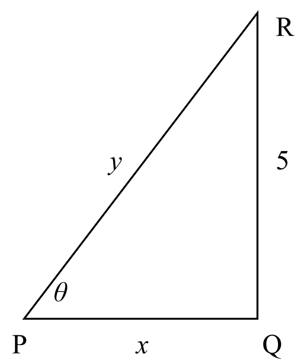
$204\pi = 72\pi + 60\pi + 9\pi\frac{dh}{dt}$

$72\pi = 9\pi\frac{dh}{dt}$

$\frac{dh}{dt} = 8$ (cm/min)

B. Paper 2 questions (LONG)

12.



(a)

Variables	Relation between the **variables**	Relation between the corresponding **rates of change**
x and θ	$\tan\theta = \frac{5}{x}$	$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = -\frac{5}{x^2}\frac{dx}{dt}$
y and θ	$\sin\theta = \frac{5}{y}$	$\cos\theta\frac{d\theta}{dt} = -\frac{5}{y^2}\frac{dy}{dt}$
x and y	$x^2 + 25 = y^2$	$2x\frac{dx}{dt} = 2y\frac{dy}{dt} \Rightarrow x\frac{dx}{dt} = y\frac{dy}{dt}$
A and x	$A = \frac{5x}{2}$	$\frac{dA}{dt} = \frac{5}{2}\frac{dx}{dt}$
P, x and y	$P = x + y + 5$	$\frac{dP}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$

(b) When $x = 5$: (i) $\theta = \frac{\pi}{4}$ (ii) $y = 5\sqrt{2}$ (iii) $A = \frac{25}{2}$ (iv) $P = 10 + 5\sqrt{2}$

(c) $\frac{dx}{dt} = 0.5$

(i) $\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = -\frac{5}{x^2}\frac{dx}{dt} \Rightarrow 4\times\frac{d\theta}{dt} = -\frac{5}{5^2}0.5 \Rightarrow \frac{d\theta}{dt} = -\frac{1}{40}$

(ii) $x\frac{dx}{dt} = y\frac{dy}{dt} \Rightarrow 5\times 0.5 = 5\sqrt{2}\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \frac{1}{2\sqrt{2}}$

(iii) $\frac{dA}{dt} = \frac{5}{2}\frac{dx}{dt} \Rightarrow \frac{dA}{dt} = \frac{5}{2}0.5 \Rightarrow \frac{dA}{dt} = \frac{5}{4}$

(iv) $\frac{dP}{dt} = \frac{dx}{dt} + \frac{dy}{dt} \Rightarrow \frac{dP}{dt} = \frac{1}{2} + \frac{1}{2\sqrt{2}} = \frac{2+\sqrt{2}}{4}$

13. It is given that $h = 2r$

(a) $V = \pi r^2 h = 2\pi r^3$

(b) $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 4\pi r^2 = 6\pi r^2$

(c) (i) $\dfrac{dV}{dt} = 6\pi r^2 \dfrac{dr}{dt}$ (ii) $\dfrac{dS}{dt} = 12\pi r \dfrac{dr}{dt}$

(d) Divide (c)(i) by (c)(ii): $\dfrac{\frac{dV}{dt}}{\frac{dS}{dt}} = \dfrac{6\pi r^2 \frac{dr}{dt}}{12\pi r \frac{dr}{dt}} = \dfrac{r}{2} \Rightarrow \dfrac{dV}{dt} = \dfrac{r}{2}\dfrac{dS}{dt}$

(e) $\dfrac{dV}{dt} = \dfrac{r}{2}\dfrac{dS}{dt} \Rightarrow 6 = \dfrac{12}{2}\dfrac{dS}{dt} \Rightarrow \dfrac{dS}{dt} = 1$